變分法簡易教程

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現代的物理學發展的框架下，喜歡從作用量 Action S出發，當物理學家寫下 Action 後（根據實驗、物理現象和限制等等猜出），針對 Action 做變分δS（Variation）後，在最小作用量原理（Principle of least action）$\delta S=0$的要求下，就可以得到物理遵守的運動方程，後續就根據不同的物理系統求滿足運動方程的物理量變化。

古典力學的發展上，由 d' Alembert 利用虛功原理可以得到

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i}-\frac{\partial L}{\partial x_i}=0$$

後來 Hamilton 進一步闡明上述方程是滿足

$$S=\int_a^b L(x_1 \dots x_i, \dot{x}_1 \dots \dot{x}_i, t)dt$$

利用變分法取極值$\delta S=0$的必然結果。

在物理學上，變分法通常（？）用於新理論在初期發展時，由於物理學家還不知道正確的運動方程，故會根據實驗成果、物理經驗等去猜 Action 可能的形式，然後利用變分法得到描述 Lagrangian 滿足的運動方程，此後變分法便功成身退，後續的解題過程只要處理運動方程即可。例如

$$S=\int_a^b L(x,\dot{x},t)dt$$

用變分後可以得到 Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}-\frac{\partial L}{\partial x}=0$$

後續理論力學課程只要會解 Euler-Lagrange equation 就好，所以整學年的課程變分法不常出現，於是應數相關的書籍對於變分法提及就不多。故在此簡單用不是數學上嚴謹的方式稍為介紹一下變分法。

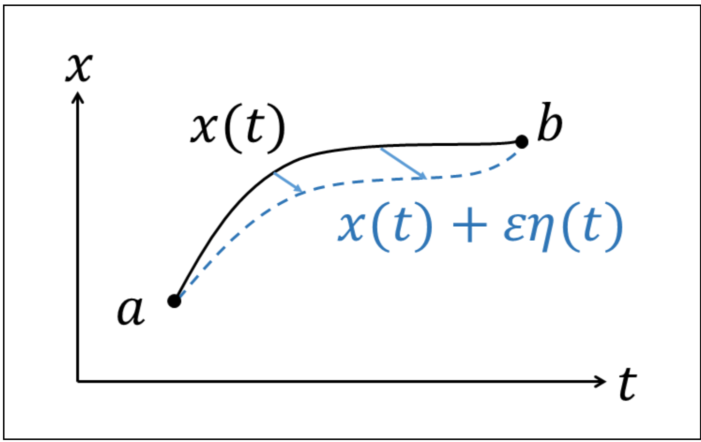
1. 第一部分：初階變分法

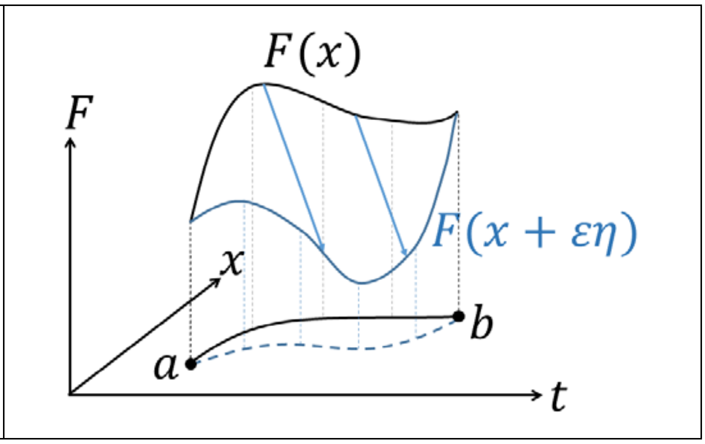
如果今天，一個泛函（Functional）的積分問題

$$S[F(x,\dot{x},t)]=\int_a^b F(x,\dot{x},t)dt$$

其中，$x=x(t)$、$\dot{x}=\frac{dx}{dt}$，固定 a、b 下改變$x(t)$的形式會對積分S造成影響，我們想找到S的極值（Extreme value，不論極大或極下）下，$x(t)$會是什麼形式？或是$F(x,\dot{x},t)$該滿足什麼條件？根據微積分的概念，在$f(x)$極值x_0附近做微小的變化$x_0+\varepsilon$時，$f(x)$是不會有變化的，即$df=0$。類似的想

法，S在極值附近時，$x(t)$稍微改變形式，$\delta S=0$。我們可以將積分問題簡單的用圖像表達，不同$x(t)$的形式表示不同的路徑連結 a 到 b 點。





如果$x(t)$是滿足S的極值，那麼如果加入微小的任意函數$\varepsilon\eta(t)$，其中$\eta(t)$滿足$\eta(a)=\eta(b)=0$，使 a、b 兩點$F(x,\dot{x},t)$不變。在加入微小的任意函數$\varepsilon\eta(t)$後

$$x(t)\to x(t)+\varepsilon\eta(t)$$
$$\dot{x}(t)\to \dot{x}(t)+\varepsilon\dot{\eta}(t)$$
$$F(x,\dot{x},t)\to F(x+\varepsilon\eta,\dot{x}+\varepsilon\dot{\eta},t)=F(x,\dot{x},t)+\delta F$$

我們知道S在極值附近，微小變化ε時，S不變：

$$\delta S=S[F(x+\varepsilon\eta,\dot{x}+\varepsilon\dot{\eta},t)]-S[F(x,\dot{x},t)]=0\leftrightarrow\frac{\delta S}{\delta\varepsilon}=0$$

δS的變化完全是由δF造成的，所以可以寫為

$$\frac{\delta S}{\delta\varepsilon}=\frac{\delta}{\delta\varepsilon}\int_a^b F dt=\int_a^b\frac{\delta F}{\delta\varepsilon}dt=0$$

利用微積分的手法，將$\frac{\delta F}{\delta\varepsilon}$展開

$$\frac{\delta S}{\delta\varepsilon}=\int_a^b\frac{\delta F}{\delta\varepsilon}dt=\int_a^b\frac{\partial F}{\partial x}\frac{\delta x}{\delta\varepsilon}+\frac{\partial F}{\partial\dot{x}}\frac{\delta\dot{x}}{\delta\varepsilon}+\frac{\partial F}{\partial t}\frac{\delta t}{\delta\varepsilon}dt$$

但我們只針對做x變分，沒有對t變分，所以

$$\frac{\delta t}{\delta\varepsilon}=0$$

$$\frac{\delta S}{\delta\varepsilon}=\int_a^b\frac{\partial F}{\partial x}\frac{\delta x}{\delta\varepsilon}+\frac{\partial F}{\partial\dot{x}}\frac{\delta\dot{x}}{\delta\varepsilon}dt$$

觀察$\frac{\delta x}{\delta\varepsilon}$、$\frac{\delta\dot{x}}{\delta\varepsilon}$：

$$\begin{cases}\dfrac{\delta x}{\delta\varepsilon}=\dfrac{\delta}{\delta\varepsilon}(x+\varepsilon\eta)=\eta\\[2ex]\dfrac{\delta\dot{x}}{\delta\varepsilon}=\dfrac{\delta}{\delta\varepsilon}(\dot{x}+\varepsilon\dot{\eta})=\dot{\eta}\end{cases}$$

可以得到

$$\frac{\delta S}{\delta\varepsilon}=\int_a^b\frac{\partial F}{\partial x}\eta+\frac{\partial F}{\partial\dot{x}}\dot{\eta}dt=\int_a^b\frac{\partial F}{\partial x}\eta+\frac{\partial F}{\partial\dot{x}}\frac{d\eta}{dt}dt=\int_a^b\frac{\partial F}{\partial x}\eta dt+\int_a^b\frac{\partial F}{\partial\dot{x}}\frac{d\eta}{dt}dt$$

我們針對最後一項做分部積分

$$\int_a^b \frac{\partial F}{\partial \dot{x}}\frac{d\eta}{dt}dt = \frac{\partial F}{\partial \dot{x}}\eta\Big|_a^b - \int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta dt$$

注意紅色這一項，因為我們要求$\eta(t)$滿足$\eta(a) = \eta(b) = 0$，所以$\frac{\partial F}{\partial \dot{x}}\eta\Big|_a^b = 0$

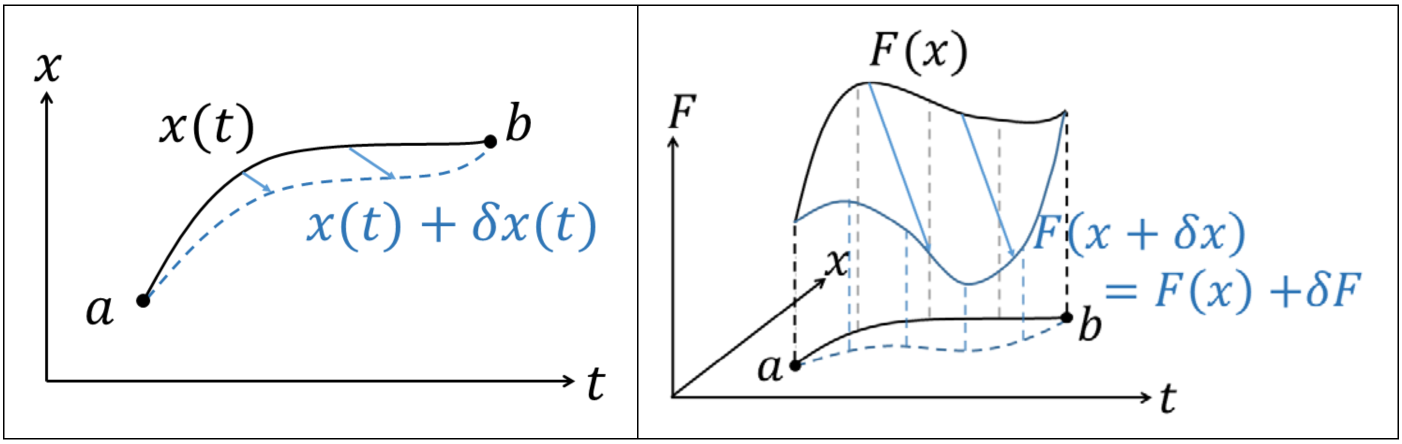
$$\therefore \frac{\delta S}{\delta \varepsilon} = \int_a^b \frac{\partial F}{\partial x}\eta dt - \int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta dt = \int_a^b \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\eta dt = 0$$

因為$\eta(t)$是任意的，所以

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \;\; Euler - Lagrange \;\; equation$$

2. 第二部分：較抽象但好用的變分法

第一部分介紹了簡單的變分法概念，但是需要引入任意的函數$\eta(t)$，手法上稍嫌煩瑣，不利於後續操作。第二部分以相同的概念，採用比較抽象的想法但相同的數學手法，演示一次變分法的操作。有點像將變分的操作類同於微分操作。



針對同一種泛函（Functional）的積分問題

$$S[F(x, \dot{x}, t)] = \int_a^b F(x, \dot{x}, t)dt$$

當我們針對x做變分，

$$x \to x + \delta x$$

變分δx滿足

$$\delta x(a) = \delta x(b) = 0$$

x的變分會導致$\dot{x}$、F發生變化

$$\begin{cases} \dot{x}(x) \to \dot{x}(x + \delta x) = \dot{x}(x) + \delta \dot{x} \\ F(x, \dot{x}, t) \to F(x + \delta x, \dot{x}(x) + \delta \dot{x}, t) = F(x, \dot{x}, t) + \delta F \end{cases}$$

我們要求$\delta S = 0$

$$\delta S = \delta \int_a^b F dt = \int_a^b \delta F dt = 0$$

概念上很好理解的是，δF的變化和δx、δx有關，所以將δF展開

$$\delta S = \int_a^b \delta F dt = \int_a^b \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial \dot{x}}\delta \dot{x} dt$$

進階：Thm.1 :微分與變分對調

如果今天微分與變分針對的對象不同，如對 t 微分$\frac{d}{dt}$、對x變分δx，則$\frac{d}{dt}$與δ可以對調。

$$\delta\dot{f} = \dot{f}(x+\delta x) - \dot{f}(x) = \frac{d}{dt}\big(f(x+\delta x) - f(x)\big) = \frac{d}{dt}\delta f$$

將$\delta\dot{x}$對調$\frac{d}{dt}\delta x$

$$\delta S = \int_a^b \frac{\partial F}{\partial x}\delta x dt + \int_a^b \frac{\partial F}{\partial \dot{x}}\left(\frac{d}{dt}\delta x\right) dt$$

同樣的手法對第二項做分部積分

$$\int_a^b \frac{\partial F}{\partial \dot{x}}\left(\frac{d}{dt}\delta x\right) dt = \frac{\partial F}{\partial \dot{x}}\delta x\Big|_a^b - \int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\delta x dt$$

注意紅色這一項，因為我們要求變分δx滿足$\delta x(a) = \delta x(b) = 0$，所以$\frac{\partial F}{\partial \dot{x}}\delta x\Big|_a^b = 0$

$$\delta S = \int_a^b \frac{\partial F}{\partial x}\delta x dt - \int_a^b \left(\frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\delta x dt = \int_a^b \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\right)\delta x dt = 0$$

$$\delta x \;\; is \;\; arbitrary.$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \;\; Euler - Lagrange \;\; equation$$

進階：Thm.2: 變分的 Chain rule

$$\delta(FG) = \frac{\partial(FG)}{\partial x}\delta x + \frac{\partial(FG)}{\partial \dot{x}}\delta\dot{x} = \left(\frac{\partial F}{\partial x}G + F\frac{\partial G}{\partial x}\right)\delta x + \left(\frac{\partial F}{\partial \dot{x}}G + F\frac{\partial G}{\partial \dot{x}}\right)\delta\dot{x}$$

$$= \left(\frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial \dot{x}}\delta\dot{x}\right)G + F\left(\frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial \dot{x}}\delta\dot{x}\right) = \delta F \cdot G + F \cdot \delta G$$

進階：Thm.3: 針對函數F 同乘同除另一函數 G，不影響變分

$$\delta F = \delta\left(F \cdot \frac{G}{G}\right) = \delta(F \cdot G \cdot G^{-1}) = \delta F \cdot G \cdot G^{-1} + F \cdot \delta G \cdot G^{-1} + F \cdot G \cdot \delta(G^{-1})$$

$$= \delta F + F \cdot \delta G \cdot G^{-1} + F \cdot G \cdot \left(-\frac{\delta G}{G^2}\right) = \delta F$$

3. 第三部分：淺談 Particle 與 EM Field 交互作用下的 Action

以前我們學古典力學時，完整描述一個 Particle 只須寫下它的 Lagrangian

$$S = \int_a^b L dt = \int_a^b T - U \;\; dt = \int_a^b T \;\; dt + \int_a^b -U \;\; dt = S_P + S_{PF}$$

其中，動能項T可視為 Free particle 的 Action S_P，位能項U就是 Particle 和 Field 交互作用的 Action S_{PF}。在電磁學我們學到 Field 也有帶有動量、能量，所以完整描述電磁運動會包含 Field 的 Action S_F

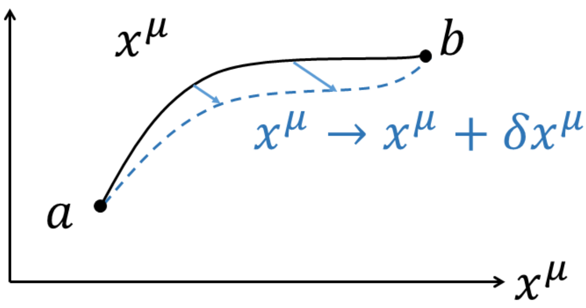
$$S = S_P + S_{PF} + S_F$$

4. 第四部分：Free Particle Action 的變分

在相對論性下，描述 Free particle 我們會利用 4-Velocity $\boldsymbol{u}=u^\mu\hat{e}_\mu=(\gamma c,\gamma\vec{v})=\frac{dx^\mu}{d\tau}\hat{e}_\mu$，其中$x^\mu$是 particle 的 4-displacement，τ是 particle 的 proper time。更進一步說$u^\mu=u^\mu(x^\nu)$，速度u^μ會隨在時空的路徑x^ν發生改變。在這邊的想法是，一個 Free particle 從時空中 a 跑到 b，我們針對不同路徑x^μ下的 Action S_P去算極值，即對x^μ作變分

$$x^\mu\rightarrow x^\mu+\delta x^\mu$$

$$\delta x^\mu(a)=\delta x^\mu(b)=0$$



Free particle 的 Action S_P為

$$S_P=\int_a^b -mc^2 d\tau$$

經過變分

$$\delta S_P=\delta\int_a^b -mc^2 d\tau=-mc^2\int_a^b \delta d\tau$$

看起來$d\tau$好像與$x^\mu\rightarrow x^\mu+\delta x^\mu$變分無關，不過回憶一件事情

$$\because c^2 d\tau^2=dx^\mu dx_\mu$$

$$\therefore c\,d\tau=\sqrt{dx^\mu dx_\mu}$$

所以

$$\delta S_P=-mc\int_a^b \delta\sqrt{dx^\mu dx_\mu}=-mc\int_a^b \frac{1}{2}\frac{\delta dx^\mu\cdot dx_\mu+dx^\mu\cdot\delta dx_\mu}{\sqrt{dx^\mu dx_\mu}}$$

進階：Thm.4: 對 Scalar 變分與上下標無關

回憶度規張量 Metric Tensor $g_{\mu\nu}$：

$$g_{\mu\nu}=\hat{e}_\mu\cdot\hat{e}_\nu$$

$$g_{\mu\nu}=g_{\nu\mu}$$

$$g^{\mu\nu}\equiv\left(g_{\mu\nu}\right)^{-1}$$

$$g^{\mu\nu}g_{\nu\omega}=\delta^\mu{}_\omega\ (Delta\ \ funcion，暫時不要跟變分\delta搞混)。$$

度規張量$g_{\mu\nu}$是時空的內稟性質(Intrinsic Property)，與x^μ無關，意思是對x^μ變分與度規$g_{\mu\nu}$無關。度規張量可以用作上下標轉換（Index lowering or raising）

$$x^\mu=g^{\mu\nu}x_\nu、x_\mu=g_{\mu\nu}x^\nu$$

因為對對x^μ變分與度規$g_{\mu\nu}$無關，所以

$$\delta x^\mu=g^{\mu\nu}\delta x_\nu、\delta x_\mu=g_{\mu\nu}\delta x^\nu$$

對一個 Scalar 作變分，例如$x^\mu y_\mu$是一個 Scalar（$x^\mu y_\mu=x_\mu y^\mu$）

$$\begin{aligned}\delta\left(x^\mu y_\mu\right)&=\delta x^\mu\cdot y_\mu+x^\mu\cdot\delta y_\mu=g^{\mu\nu}\delta x_\nu\cdot g_{\mu\omega}y^\omega+g^{\mu\nu}x_\nu\cdot g_{\mu\omega}\delta y^\omega\\&=g^{\mu\nu}g_{\mu\omega}(\delta x_\nu\cdot y^\omega+x_\nu\cdot\delta y^\omega)=g^{\nu\mu}g_{\mu\omega}\delta(x_\nu y^\omega)\\&=\delta^\nu{}_\omega\delta(x_\nu y^\omega)=\delta(x_\omega y^\omega)=\delta\left(x_\mu y^\mu\right)\end{aligned}$$

同理

$$x^\mu\delta y_\mu=x_\mu\delta y^\mu$$

所以

$$\delta dx^\mu\cdot dx_\mu+dx^\mu\cdot\delta dx_\mu=\delta dx^\mu\cdot dx_\mu+dx_\mu\cdot\delta dx^\mu=2\delta dx^\mu\cdot dx_\mu$$

$$S_P = -mc\int_a^b \frac{1}{2}\frac{2\delta dx^\mu \cdot dx_\mu}{\sqrt{dx^\mu dx_\mu}} = -mc\int_a^b \frac{\delta dx^\mu \cdot dx_\mu}{cd\tau} = -m\int_a^b \delta dx^\mu \cdot u_\mu$$

利用 *Thm.1* 的方法，我們將δdx^μ對調成$d\delta x^\mu$，並作分部積分

$$S_P = -m\int_a^b u_\mu d\delta x^\mu = -u_\mu \delta x^\mu\Big|_a^b + m\int_a^b du_\mu \delta x^\mu$$

邊界項因為變分邊界$\delta x^\mu(a) = \delta x^\mu(b) = 0$，後面那一項利用 *Thm.3* 同乘同除$d\tau$不影響變分

$$S_P = m\int_a^b du_\mu \delta x^\mu = \int_a^b m\frac{du_\mu}{d\tau}\delta x^\mu d\tau$$

所以對 Free particle 而言，$\delta S_P = 0$使得

$$m\frac{du_\mu}{d\tau} = 0$$

觀察$\mu = 1\sim 3$

$$m\frac{d\vec{v}}{d\tau} = 0$$

Free particle 沒有加速度，保持等速運動。

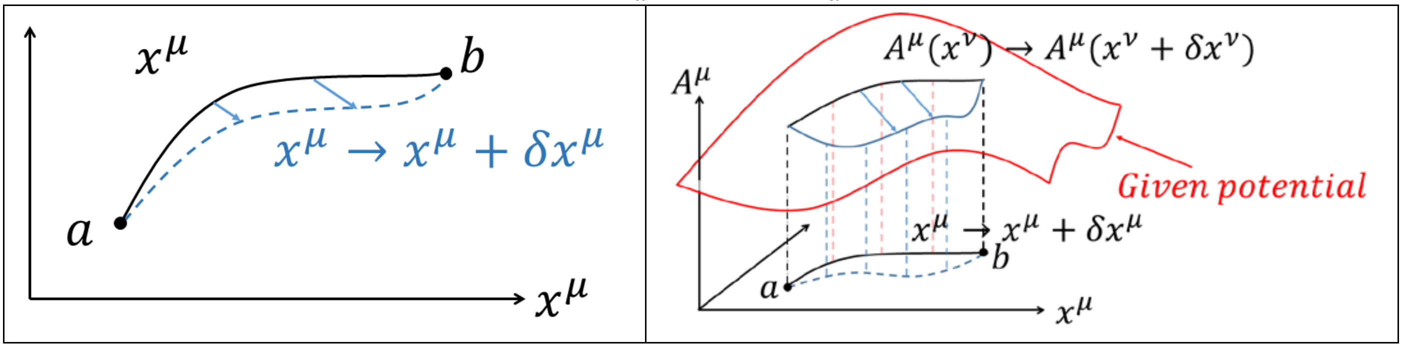
5. 第五部分：Charge Particle 與 EM Field 作用下的變分與運動方程

在這一部分中，我們想探討一個 Charge Particle 在給定的 EM Field 下如何運動？（注意喔，給定的 EM Field 表示我們不對 EM Field 作變分）。相對論性電磁學下我們會寫下 4-Potential $\boldsymbol{A} = A^\mu \hat{e}_\mu = (\phi, \vec{A})$，在這邊採用高斯制（Gaussian unit），而 Charge Particle 交互作用的 Action S_{PF}會寫成

$$S_{PF} = \int_a^b -\frac{e}{c}A_\mu dx^\mu$$

完整的描述 Charge Particle 運動即為

$$S = S_P + S_{PF} = \int_a^b -mc^2 d\tau + \int_a^b -\frac{e}{c}A_\mu dx^\mu$$



在這邊，我們考慮 Charge Particle 在時空中的路徑作x^μ變分

$$x^\mu \to x^\mu + \delta x^\mu$$

雖然我們沒有對 EM Field A^μ作變分，但是走不同路徑感受到的位能是不一樣的，所以 Action 走不同的路徑會有不同的A^μ（意思是A^μ的變化來自於路徑x^μ不同，而不是對A^μ作變分）

$$A^\mu(x^\mu) \to A^\mu(x^\mu + \delta x^\mu) = A^\mu(x^\mu) + \delta A^\mu$$

我們計算δS_{PF}如何變分：

$$\delta S_{PF} = \delta\int_a^b -\frac{e}{c}A_\mu dx^\mu = -\frac{e}{c}\int_a^b \delta A_\mu \cdot dx^\mu - \frac{e}{c}\int_a^b A_\mu \cdot \delta dx^\mu$$

利用 *Thm.1* 的方法，我們將δdx^μ對調$d\delta x^\mu$

$$\delta dx^\mu = d\delta x^\mu$$

並作分部積分，邊界項會消失

$$\delta S_{PF} = -\frac{e}{c}\int_a^b \delta A_\mu \cdot dx^\mu - \frac{e}{c}\int_a^b A_\mu \cdot d\delta x^\mu = -\frac{e}{c}\int_a^b \delta A_\mu \cdot dx^\mu - \frac{e}{c}A_\mu \cdot \delta x^\mu\Big|_a^b + \frac{e}{c}\int_a^b dA_\mu \cdot \delta x^\mu$$

利用 *Thm.3* 同乘同除$d\tau$不影響變分

$$\delta S_{PF} = -\frac{e}{c}\int_a^b \delta A_\mu \cdot dx^\mu + \frac{e}{c}\int_a^b dA_\mu \cdot \delta x^\mu = -\frac{e}{c}\int_a^b \delta A_\mu \cdot \frac{dx^\mu}{d\tau}d\tau + \frac{e}{c}\int_a^b \frac{dA_\mu}{d\tau}d\tau \cdot \delta x^\mu$$

因為

$$\begin{cases} \delta A_\mu = \dfrac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \\ \dfrac{dA_\mu}{d\tau} = \dfrac{\partial A_\mu}{\partial x^\nu}\dfrac{dx^\nu}{d\tau} = \dfrac{\partial A_\mu}{\partial x^\nu}u^\nu \end{cases}$$

代入得到

$$\begin{aligned} \delta S_{PF} &= -\frac{e}{c}\int_a^b \frac{\partial A_\mu}{\partial x^\nu}\delta x^\nu \cdot u^\mu d\tau + \frac{e}{c}\int_a^b \frac{\partial A_\mu}{\partial x^\nu}u^\nu d\tau \cdot \delta x^\mu \\ &= -\frac{e}{c}\int_a^b (\partial_\nu A_\mu)u^\mu \delta x^\nu d\tau + \frac{e}{c}\int_a^b (\partial_\nu A_\mu)u^\nu \delta x^\mu d\tau \end{aligned}$$

我們想要把變分δx^ν和δx^μ一起提出來，但是上標不一樣。但因為每一項μ、ν都是 Dummy index，可以互換$\mu \leftrightarrow \nu$，我們把第一項的μ、ν互換，就可以把兩項合併

$$\begin{aligned} \delta S_{PF} &= -\frac{e}{c}\int_a^b (\partial_\mu A_\nu)u^\nu \delta x^\mu d\tau + \frac{e}{c}\int_a^b (\partial_\nu A_\mu)u^\nu \delta x^\mu d\tau \\ &= -\frac{e}{c}\int_a^b (\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu \delta x^\mu d\tau \end{aligned}$$

完整考慮 Charge Particle 在 EM Field 中的運動

$$\delta S = \delta S_P + \delta S_{PF} = 0$$

所以

$$\begin{aligned} \delta S_P + \delta S_{PF} &= \int_a^b m\frac{du_\mu}{d\tau}\delta x^\mu d\tau - \frac{e}{c}\int_a^b (\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu \delta x^\mu d\tau \\ &= \int_a^b \left[m\frac{du_\mu}{d\tau} - \frac{e}{c}(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu\right]\delta x^\mu d\tau = 0 \end{aligned}$$

會得到

$$m\frac{du_\mu}{d\tau} - \frac{e}{c}(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu = 0$$

$$m\frac{du_\mu}{d\tau} = \frac{e}{c}(\partial_\mu A_\nu - \partial_\nu A_\mu)u^\nu \equiv \frac{e}{c}F_{\mu\nu}u^\nu$$

我們定義電磁張量 Electromagnetic Tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

電場$\vec{E}$、磁場$\vec{B}$與ϕ、$\vec{A}$的關係（高斯制）

$$\begin{cases} \vec{E} = -\nabla\phi - \dfrac{1}{c}\dfrac{\partial\vec{A}}{\partial t} \\ \vec{B} = \nabla\times\vec{A} \end{cases}$$

其中

$$A_\nu \to (\phi, -A_x, -A_y, -A_z)$$

$$\partial_\mu \to \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial ct}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

可以計算

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

6. 第六部分：4-Volume d^4x與 Lagrangian Density $\mathcal{L}$

因為時空在相對論下是等價的，利用相對性原理（Principle of Relativity）和最小作用量原理（Principle of least action）的要求，物理學家會將 Action S寫成 Scalar 的形式，從而保證在任何作標系下$\delta S = 0$。剛剛我們所列下來的 Action：

$$S_P = -mc^2\int_a^b d\tau$$

$$S_{PF} = -\frac{e}{c}\int_a^b A_\mu dx^\mu$$

$$S = S_P + S_{PF} = -mc^2\int_a^b d\tau - \frac{e}{c}\int_a^b A_\mu dx^\mu = \int_a^b -\gamma mc^2 - \gamma e\phi + \frac{e}{c}\vec{A}\cdot\gamma\vec{v}\ \ dt = \int_a^b L dt$$

雖然 Action 都滿足 Scalar 的要求，但是 Lagrangian L本身並不是 Scalar，因為換到不同座標系下會不一樣，物理學家於是想要進一步將 Lagrangian L改寫成 Scalar 的形式。我們定義 4-Volume $d^4x = dc\tau dV = dc\tau dxdydz$，並將原本的 Action 改寫

$$S_P = -mc^2\int_a^b d\tau = -\int \rho_m dV\, c^2\int_a^b d\tau = -\iint \rho_m c\, dc\tau dV = \int -\rho_m c\, d^4x$$

$$S_{PF} = -\frac{e}{c}\int_a^b A_\mu dx^\mu = -\frac{\int \rho dV}{c}\int_a^b A_\mu \frac{dx^\mu}{d\tau}d\tau = -\frac{1}{c}\iint \rho A_\mu u^\mu d\tau dV = -\frac{1}{c^2}\iint A_\mu J^\mu dc\tau d^4x$$

$$= -\frac{1}{c^2}\int A_\mu J^\mu d^4x$$

其中，4-current density $\boldsymbol{J} = J^\mu\hat{e}_\mu = \rho u^\mu\hat{e}_\mu$。特別的是，$d^4x$是一個不變量 Invariant，所以是一個 Scalar。另外在加上 EM Field 的 Action S_F

$$S_F = -\frac{1}{16\pi c}\int F_{\mu\nu}F^{\mu\nu}d^4V$$

$$S = S_P + S_{PF} + S_F = \int -\rho_m c - \frac{1}{c^2}A_\mu J^\mu - \frac{1}{16\pi c}F_{\mu\nu}F^{\mu\nu}\ \ d^4V \equiv \int \mathcal{L}d^4V$$

因為 4-Volume d^4x是一個 Scalar，Action 也是一個 Scalar，所以$\mathcal{L}$也是一個 Scalar。$\mathcal{L}$我們稱為 Lagrangian density，Lagrangian density　$\mathcal{L}$在任何座標系下都是 Scalar，形式保持不變：

$$\mathcal{L} = -\rho_m c - \frac{1}{c^2} A_\mu J^\mu - \frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu}$$

進階：d^4x是一個不變量 Invariant

物理 proof

因為 Time dilation　和　Length contraction 相反。如果τ、$\bar{x}$是 proper time 和 proper length

$$t = \gamma\tau$$

$$x = \frac{\bar{x}}{\gamma}$$

所以

$$dctdx = dc(\gamma\tau)d\left(\frac{\bar{x}}{\gamma}\right) = dc\tau d\bar{x}$$

數學 proof

回憶 Jacobian　J

$$dxdy = rdrd\theta = J(r,\theta)drd\theta$$

其中

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial r cos\theta}{\partial r} & \frac{\partial r cos\theta}{\partial \theta} \\ \frac{\partial r sin\theta}{\partial r} & \frac{\partial r sin\theta}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix} = r$$

同理

$$dc\bar{t}d\bar{x} = J(ct,x)dctdx = \begin{vmatrix} \frac{\partial c\bar{t}}{\partial ct} & \frac{\partial c\bar{t}}{\partial x} \\ \frac{\partial \bar{x}}{\partial ct} & \frac{\partial \bar{x}}{\partial x} \end{vmatrix} dctdx$$

回憶勞倫茲轉換

$$\begin{cases} c\bar{t} = \gamma(ct - \beta x) \\ \bar{x} = \gamma(x - \beta ct) \end{cases}$$

代回去計算

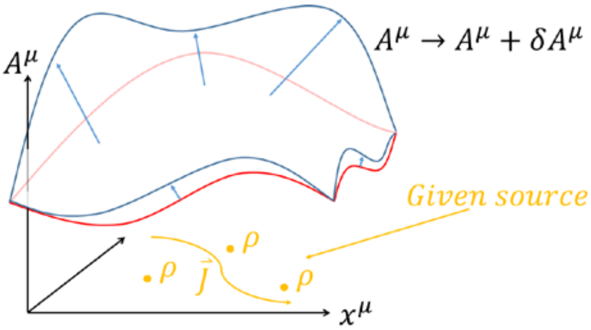
$$dc\bar{t}d\bar{x} = \begin{vmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{vmatrix} dctdx = \gamma^2(1-\beta^2)dctdx = dctdx$$

所以d^4x在勞倫茲轉換下是一個不變量。

7. 第七部分：給定 Source 下對 EM Field 變分與 Maxwell eq

在這一部分，我們在給定 Source 下，討論 EM Field 的分佈（注意喔，給定 Source 下表示我們不對 Source 下的分佈x^μ作變分）。

$$S = S_{PF} + S_F = \int -\frac{1}{c^2} A_\mu J^\mu d^4x + \int -\frac{1}{16\pi c} F_{\mu\nu} F^{\mu\nu} d^4V$$



我們想知道 EM Field 的分佈，所以我們針對A^μ作變分

$$A^\mu \to A^\mu + \delta A^\mu$$

我們來觀察J^μ、d^4x、$F^{\mu\nu}$會不會受到影響？

$$J^\mu = \rho \frac{dx^\mu}{d\tau} = J^\mu(x^\nu)$$

$$d^4x = d^4x(x^\nu)$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = F^{\mu\nu}(A^\omega)$$

可見只有電磁張量$F^{\mu\nu}$會受到A^μ的變分$A^\mu \to A^\mu + \delta A^\mu$有關

$$F^{\mu\nu}(A^\omega) \to F^{\mu\nu}(A^\omega + \delta A^\omega) = F^{\mu\nu}(A^\omega) + \delta F^{\mu\nu}$$

所以S_{PF}的變分很簡單

$$\delta S_{PF} = \delta \int -\frac{1}{c^2} A_\mu J^\mu d^4V = -\frac{1}{c^2} \int \delta A_\mu \cdot J^\mu d^4V$$

至於S_F的變分就稍嫌複雜

$$S_F = -\frac{1}{16\pi c} \int F_{\mu\nu} F^{\mu\nu} d^4V$$

$$\delta S_F = -\frac{1}{16\pi c} \delta \int F_{\mu\nu} F^{\mu\nu} d^4V = -\frac{1}{16\pi c} \int \delta F_{\mu\nu} \cdot F^{\mu\nu} + F_{\mu\nu} \cdot \delta F^{\mu\nu} d^4V$$

利用 *Thm.4:* 對 Scalar 變分與上下標無關，所以$\delta F_{\mu\nu} \cdot F^{\mu\nu} = F_{\mu\nu} \cdot \delta F^{\mu\nu}$，會有兩倍

$$\delta S_F = -\frac{1}{16\pi c} \int 2\delta F_{\mu\nu} \cdot F^{\mu\nu} d^4V = -\frac{1}{8\pi c} \int \delta F_{\mu\nu} \cdot F^{\mu\nu} d^4V$$

$$= -\frac{1}{8\pi c} \int \delta(\partial_\mu A_\nu - \partial_\nu A_\mu) \cdot F^{\mu\nu} d^4V$$

$$= -\frac{1}{8\pi c} \int \delta(\partial_\mu A_\nu) \cdot F^{\mu\nu} d^4V + \frac{1}{8\pi c} \int \delta(\partial_\nu A_\mu) \cdot F^{\mu\nu} d^4V$$

因為每一項μ、ν都是 Dummy index，可以互換$\mu \leftrightarrow \nu$，我們把第一項的μ、ν互換，就可以把兩項合併

$$\delta S_F = -\frac{1}{8\pi c} \int \delta(\partial_\nu A_\mu) \cdot F^{\nu\mu} d^4V + \frac{1}{8\pi c} \int \delta(\partial_\nu A_\mu) \cdot F^{\mu\nu} d^4V$$

$$= \frac{1}{8\pi c} \int \delta(\partial_\nu A_\mu) \cdot (-F^{\nu\mu} + F^{\mu\nu}) d^4V$$

回憶電磁張量

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$F^{\mu\nu}$是一個反對稱張量，所以$F^{\nu\mu} = -F^{\mu\nu}$

代回去會多兩倍

$$\delta S_F = \frac{1}{8\pi c}\int \delta(\partial_\nu A_\mu)\cdot(F^{\mu\nu}+F^{\mu\nu})d^4V = \frac{1}{4\pi c}\int \delta(\partial_\nu A_\mu)\cdot F^{\mu\nu}d^4V$$

利用 *Thm.1* 的方法，我們將δ和∂_ν對調

$$\delta(\partial_\nu A_\mu) = \partial_\nu(\delta A_\mu)$$

$$\delta S_F = \frac{1}{4\pi c}\int \partial_\nu(\delta A_\mu)\cdot F^{\mu\nu}d^4V$$

利用微分的 Chain rule，

$$\partial_\nu(\delta A_\mu)\cdot F^{\mu\nu} = \partial_\nu(\delta A_\mu\cdot F^{\mu\nu}) - \delta A_\mu\cdot\partial_\nu(F^{\mu\nu})$$

將積分拆成兩項

$$\delta S_F = \frac{1}{4\pi c}\int \partial_\nu(\delta A_\mu\cdot F^{\mu\nu})d^4V - \frac{1}{4\pi c}\int \delta A_\mu\cdot\partial_\nu(F^{\mu\nu})d^4V$$

回憶 Divergence theorem

$$\int \nabla\cdot\vec{F}dV = \oint \vec{F}\cdot d\vec{S}$$

一個體積分，可以改寫成對體表面的面積分

寫成 Levi-Civita symbol

$$\int \partial_\nu F^\nu dV = \oint F^\nu dS_\nu$$

所以第一項利用 Divergence theorem，但是 Boundary 上的$\delta A_\mu = 0$，所以

$$\frac{1}{4\pi c}\int \partial_\nu(\delta A_\mu\cdot F^{\mu\nu})d^4V = \frac{1}{4\pi c}\oint \delta A_\mu\cdot F^{\mu\nu}dS_\nu = 0$$

所以

$$\delta S_F = -\frac{1}{4\pi c}\int \partial_\nu(F^{\mu\nu})\cdot\delta A_\mu d^4V$$

合併$\delta S_{PF}+\delta S_F$

$$\delta S_{PF}+\delta S_F = -\frac{1}{c^2}\int J^\mu\cdot\delta A_\mu d^4V - \frac{1}{4\pi c}\int \partial_\nu(F^{\mu\nu})\cdot\delta A_\mu d^4V$$

$$= \int\left[-\frac{1}{c^2}J^\mu - \frac{1}{4\pi c}\partial_\nu F^{\mu\nu}\right]\delta A_\mu d^4V = 0$$

會得到

$$-\frac{1}{c^2}J^\mu - \frac{1}{4\pi c}\partial_\nu F^{\mu\nu} = 0$$

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c}J^\mu$$

$$\textit{Maxwell eq (Gaussian Unit)}\begin{cases}\nabla\cdot\vec{E} = 4\pi\rho\\ \nabla\cdot\vec{B} = 0\\ \nabla\times\vec{E} = -\dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t}\\ \nabla\times\vec{B} = \dfrac{4\pi}{c}\vec{J} + \dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t}\end{cases}$$

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c}J^\mu \rightarrow \begin{cases} \nabla\cdot\vec{E} = 4\pi\rho \\ \nabla\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} \end{cases}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

電磁張量有特別的關係式

$$\partial_\omega F_{\mu\nu} + \partial_\mu F_{\nu\omega} + \partial_\nu F_{\omega\mu} = 0$$

展開

$$\partial_\omega(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\omega - \partial_\omega A_\nu) + \partial_\nu(\partial_\omega A_\mu - \partial_\mu A_\omega) = 0$$

$$\partial_\omega\partial_\mu A_\nu - \partial_\omega\partial_\nu A_\mu + \partial_\mu\partial_\nu A_\omega - \partial_\mu\partial_\omega A_\nu + \partial_\nu\partial_\omega A_\mu - \partial_\nu\partial_\mu A_\omega = 0$$

這條關係式會得到

$$\partial_\omega F_{\mu\nu} + \partial_\mu F_{\nu\omega} + \partial_\nu F_{\omega\mu} = 0 \rightarrow \begin{cases} \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \end{cases}$$